The Closed form Solution of the Standard Ramsey Growth Model with CRRA Consumer Preferences and Logistic Growth of Consumption per Capital*

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Abstract

In order to find a closed form solution of the Ramsey growth model usually author's take consumer preferences and production technology as given. Especially with the assumptions of consumer CRRA preferences and Cobb-Douglas production technology Smith (2006) derived the widely adopted solution in case of capital's share equals consumer's risk aversion parameter, which implies consumption per capital to be constant. We skip the assumption of a given production technology and replace this by the assumption that consumption per capital follows a logistic growth process. In this case we derive the general solution, for the evolution of capital and consumption in time. Not surprisingly this includes the solution formerly described. But additionally, at least in a technical way, we obtain a closed form solution with a non linear dependence between consumption and capital.

Keywords: Ramsey Growth Model, Bernoulli Differential equation,
Hypergeometric Functions
JEL Classification: O40, C63, C68

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1 Introduction

The Ramsey growth model (Ramsey (1928)), further elaborated by Cass (1965) and Koopmans (1963) is one of the most examined and best understood models in economics and can be found in almost every standard textbook of macroeconomic theory (i.e. Barro and Sala-i-Martin (2003), Blanchard and Fischer (1989) or Romer (2005)).

Not surprisingly the general formulation of the Ramsey growth model exhibits no explicit solution, since consumer preferences and production technology are not further specified, except for assumptions about monotonicity and curvature of the functions and constant economies of scale in production. But even if consumer preferences and production technology are explicitly defined closed form solutions are only known under further restrictions such as a constant gross savings rate or a hundred percent depreciation on capital.[1] For an excellent survey see Smith (2006). In particular Smith (2006) provides a closed form solution for production technology to be Cobb Douglas, consumer preferences to be CRRA and consumer's risk aversion parameter equals capital's share.[2] This is shown by converting the Ramsey differential equations via Bernoulli transformation into a simple autonomous logistic equation, which is a special case of a Bernoulli differential equation. Although this parameter setting is widely adopted in the literature[3] we think, we can give a more general answer to the question of the existence of a closed form solution of the Ramsey growth model. In order to tackle the problem, in some sense we follow an inverse approach similarily to Chang (1988). In Chang (1988) production technology is fixed to be Cobb-Douglas in order to derive consumer preferences, if consumption follows some standard path. We take the other part as fixed starting with consumer preferences to be CRRA and ask which production technology satisfies the Ramsey model, if consumption per capital follows a logistic growth process.

As we can show the whole problem reduces to the task to sequentially solve some Bernoulli differential equations, whereby in the last step we use a standard transformation in order to obtain a hypergeometric function. In our eyes this simplifies the mathematics in some sense compared to Smith (2006), Boucekkine and Ruiz-Tamarit (2008), Scarpello and Ritelli (2003) or Germanà and Guerrini (2005) who apply also hypergeometric functions. It turns out, that the production function necessarily consists of a Cobb-Douglas part while in a wide range of parameter settings the other part must be linear in capital, which follows from the constancy of consumption per capital implied by the transversality condition. This repeats in a general way the result of Smith (2006). But additionally in other parameter constellations we obtain a non linear dependence between consumption and capital.

[1] An exception constitutes the approach of Mehlum (2005), who defines only production technology to be Leontief and consumer preferences to be CRRA for the calculation of a closed form solution.

[2] Extending the Cobb Douglas production function with a linear term in capital a closed from solution still exists, as shown in Smith (2006) too.

[3] For an overview see Feicht and Stummer (2010))

2 The Model

The standard Ramsey growth model with CRRA consumer preferences and constant returns to scale production technology is characterized by the following differential equations including the transversality condition (TVC):

$$\begin{aligned} \tfrac{\dot{c}}{c} &= \tfrac{1}{\theta}(f'(k) - \delta - \rho) && (1) \\ \tfrac{\dot{k}}{k} &= \tfrac{f(k)}{k} - \delta - \tfrac{c}{k} && (2) \\ \lim_{t\to\infty} e^{-\rho t} c^{-\theta} k &= 0 \quad (TVC) && (3) \end{aligned}$$

with consumption c, capital k and production $f(k)$ in effective terms,[4] depreciation of capital $\delta \in [0, 1]$, consumer's risk aversion parameter $\theta \in [0, 1]$, time preference rate $\rho > 0$ and time $t \geq 0$.

In the following we define consumption per capital by $\mathcal{C} := \frac{c}{k}$ which implies

$$\dot{\mathcal{C}} = \left(\frac{\dot{c}}{c} - \frac{\dot{k}}{k}\right)\mathcal{C} \tag{4}$$

Multiplying equations (1) and (2) by $\mathcal{C}$, then subtraction together with equation (4) leads to

$$\dot{\mathcal{C}} + \left(\frac{f(k)}{k} - \delta - \frac{1}{\theta}(f'(k) - \delta - \rho)\right)\mathcal{C} = \mathcal{C}^2 \tag{5}$$

Additionally the transversality condition then reads

$$\lim_{t\longrightarrow\infty} e^{-\rho t}\mathcal{C}^{-\theta}k^{1-\theta} = 0 \tag{6}$$

Defining

$$\mathcal{M}(k(t)) := \frac{f(k)}{k} - \delta - \frac{1}{\theta}(f'(k) - \delta - \rho) = \mathcal{C} - \frac{\dot{\mathcal{C}}}{\mathcal{C}} \tag{7}$$

we observe, that equation (5) represents a Bernoulli differential equation:[5]

$$\dot{\mathcal{C}} + \mathcal{M}(k(t))\mathcal{C} = \mathcal{C}^2 \tag{8}$$

[4] For simplicity we imputed no labor growth, but our results hold also for a constant growth rate of labor

[5] Recapulate, that the Bernoulli differential equation is given by

$$\frac{dy}{dt} + P(t)y = Q(t)y^n$$

with the general solution

Since $\mathcal{M}(t)$ is still an arbitrary function, we need a further assumption in order to make the differential equation (8) directly tractable. Therefore we assume the Ramsey model to be recursive in the sense, that the ratio of consumption per capital $\mathcal{C}$ follows a logistic growth process.[6]

$$\dot{\mathcal{C}} + A\mathcal{C} = B\mathcal{C}^2 \tag{9}$$

with some constants[7] A and B $(A, B \neq 0)$.

Since the logistic differential equation is a special Bernoulli differential equation from footnote 5 we obtain the general the solution of equation (9)

$$\mathcal{C}(t) = \frac{\frac{A}{B}}{1 - C_1 e^{At}} \tag{10}$$

with some constant C_1. Since equation (10) has only a meaningful economic interpretation if $\mathcal{C}(t) \geq 0$ for all $t \geq 0$, the following proposition 1 shows the resulting parameter restrictions for A, B and C_1:

Proposition 1
Suppose $\mathcal{C}(t) = \frac{\frac{A}{B}}{1-C_1 e^{At}} \geq 0$ *for all* $t \geq 0$ *then*

(i) $C_1 \leq 0 \quad A > 0 \quad B > 0$

(ii) $C_1 < 1 \quad A < 0 \quad B < 0$

(iii) $C_1 > 1 \quad A > 0 \quad B < 0$

Proof :
Inserting every parameter constellation other than $(i), (ii)$ or (iii) $\mathcal{C}(t)$ is either negative or there exists a $t' > 0$ such that $C_1 e^{At'} = 1$.

■

$$y(t) = \left[\frac{(1-n)\int e^{(1-n)\int P(t'')dt''} Q(t')dt' + const.}{e^{(1-n)\int P(t'')dt''}} \right]^{\frac{1}{1-n}}$$

with $P(t)$ and $Q(t)$ some arbitrary functions and $n \neq 1$.

[6]Smith (2006) supposes that the Ramsey model is only recursive, if production technology is Cobb-Douglas and capital's share equals the consumer's risk parameter. Actually he showed this also for a Cobb-Douglas production function extended by a linear term in capital. We show, that recursivity can be reached in a even more general way.

[7]Note that for standard logistic growth we have $A, B < 0$. But in the widely adopted case of CRRA consumer preferences, Cobb-Douglas production technology and capital's share equals the coefficient of consumer's risk aversion $A, B > 0$ (see Smith (2006)). We show, that there is at least an academic solution for $A, B < 0$ too.

By inserting equation (9) into equation (7) the resulting equation can also be interpreted as a Bernoulli differential equation for $f(k)$ with

$$f'(k) - \frac{\theta}{k} f(k) = \left[(\delta(1-\theta) - (A\theta - \rho) + \theta(B-1)\mathcal{C}(t) \right] \left[f(k) \right]^0 \tag{11}$$

This we can directly integrate for the production technology $f(k)$

$$f(k) = C_2 k^\theta + \left[\delta - \frac{A\theta - \rho}{1-\theta} + \frac{\theta(B-1)}{1-\theta} \mathcal{C}(t) \right] k \tag{12}$$

with some constant $C_2 \geq 0$ due to economic relevancy. As we can already see CRRA consumer preferences imply that production technology splits into a Cobb-Douglas part and a linear part in capital k if $\mathcal{C}$ is constant or $B = 1$. Together this represents an Ak-type production technology introduced by Jones and Manuelli (1990) and Jones and Manuelli (1997). Inserting equation (12) into the capital accumulation equation (2) we obtain

$$\dot{k} + \underbrace{\left[\frac{A\theta - \rho}{1-\theta} - \frac{B\theta - 1}{1-\theta} \mathcal{C}(t) \right]}_{P(t)} k = \underbrace{C_2}_{Q(t)} k^{\overbrace{\theta}^{n}} \tag{13}$$

which is again a Bernoulli differential equation.[8]

From equation (10) and footnote 5 we obtain

$$e^{(1-\theta)\int P(t)dt} = e^{\int A\theta - \rho - \frac{(\theta - \frac{1}{B})A}{1 - C_1 e^{At}} dt} = C_3 \left(C_1 e^{At} \right)^{(\frac{1}{B} - \frac{\rho}{A})} (1 - C_1 e^{At})^{(\theta - \frac{1}{B})} =: X(t) \tag{14}$$

with some constant C_3. For the evolution of capital $k(t)$ in time together with the standard substitution $\tau\zeta = C_1 e^{At'}$ and $\zeta = C_1 e^{At}$ implying $d\tau = A\tau dt'$, we obtain:

$$\begin{aligned} k(t) &= \left[\frac{(1-\theta)\int_0^t Q(t')X(t')dt' + C_4'}{X(t)} \right]^{\frac{1}{1-\theta}} \\ &= \left[C_2 \frac{(1-\theta)}{A(1-\zeta)^{(\theta - \frac{1}{B})}} \int_0^1 \frac{\tau^{(\frac{1}{B} - \frac{\rho}{A} - 1)}(1-\tau)^0}{(1-\tau\zeta)^{(\frac{1}{B} - \theta)}} d\tau + \frac{C_4''}{\zeta^{(\frac{1}{B} - \frac{\rho}{A})}(1-\zeta)^{(\theta - \frac{1}{B})}} \right]^{\frac{1}{1-\theta}} \end{aligned} \tag{15}$$

with some constants C_4' and C_4''. Finally with the definition of hypergeometric functions[9]

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[8] Note the interesting feature, that capital accumulation does not directly depend anymore on capital depreciation δ. However a dependence is still possible via the constants A and B within the formula for the ratio of consumption per capital $\mathcal{C}$.

[9] Please do not confuse c for consumption and c as a parameter of the general hypergeometric function in the following. For an introduction in hypergeometric functions see Lebedev (1972), Andrews, Askey, and Roy (2001) and Luke (1969). For a quick look in the worldwideweb see

$$
{}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-tz)^a}dt = (1-z)^{-a}{}_2F_1(a,c-b;c;\tfrac{z}{z-1}) \quad (16)
$$

and the functional equation $\frac{\Gamma(x+1)}{\Gamma(x)} = x$ of the Gamma function we end up with

$$
k(t) \;=\; \left[C_2\frac{(1-\theta)B}{A-B\rho}{}_2F_1\left(\tfrac{1}{B}-\theta,1;\tfrac{1}{B}-\tfrac{\rho}{A}+1;\tfrac{C_1e^{At}}{C_1e^{At}-1}\right)+\frac{C_4}{(e^{At})^{\frac{1}{B}-\frac{\rho}{A}}(1-C_1e^{At})^{\theta-\frac{1}{B}}}\right]^{\frac{1}{1-\theta}} \quad (17)
$$

with some constant C_4.

Observe that for case (iii) in proposition 1 $(1-C_1e^{At})^{(\theta-\frac{1}{B})} \notin \mathbb{R}$. Therefore we rule out this case in the following. Equation (17) together with equation (10) principally define also the consumption path $c(t)$, but it is left to evaluate the transversality condition. It turns out that for some parameter constellations this simplifies the solution drastically, since TVC implies $\mathcal{C} = const.$ over time. The mathematics repeats more or less Smith (2006) in a slightly more general sense:

Inserting equation (17) and equation (10) into TVC and after collecting the parameters we obtain

$$
\lim_{t\to\infty}\left(\frac{A}{B}\right)^{-\theta}\left[\overbrace{C_4(e^{-At}-C_1)^{\frac{1}{B}}}^{T_1}+\overbrace{C_2\tfrac{(1-\theta)(A-B\rho)}{A^2B}{}_2\tilde{F}_1(t)\,e^{-\rho t}(1-C_1e^{At})^{\theta}}^{T_2}\right]=0 \quad (18)
$$

with

$$
{}_2\tilde{F}_1(t) = {}_2F_1\left(\tfrac{1}{B}-\theta,1;\tfrac{1}{B}-\tfrac{\rho}{A}+1;\tfrac{C_1e^{At}}{C_1e^{At}-1}\right) \quad (19)
$$

The implications of TVC for $k(t)$ are summarized in the following proposition

Proposition 2

(i) Suppose $C_1 \leq 0 \quad A > 0$ *and* $B > 0$,

$$
k(t) = \left[C_2\tfrac{(1-\theta)B}{A-B\rho}{}_2\tilde{F}_1(t)+\frac{C_4}{(e^{At})^{\left(\frac{1}{B}-\frac{\rho}{A}\right)}(1-C_1e^{At})^{\theta-\frac{1}{B}}}\right]^{\frac{1}{1-\theta}}
$$

and

$$
\mathcal{C}(t) = \frac{\frac{A}{B}}{1-C_1e^{At}}
$$

then

http://dlmf.nist.gov/ and http://mathworld.wolfram.com/HypergeometricFunction.html. For applications in economics including similar problems than ours see Abadir (1999), Boucekkine and Ruiz-Tamarit (2008), Smith (2006), Germanà and Guerrini (2005), Scarpello and Ritelli (2003) and Hiraguchi (2009).

(ii) Suppose $C_1 < 1 \quad A < 0$ and $B < 0$, then there exist C_1, A and B such that TVC is fulfilled without $\mathcal{C}(t) = const.$.

Proof :
We rewrite

$$e^{-\rho t}\mathcal{C}^{-\theta}k^{1-\theta} = \left(\frac{A}{B}\right)^{-\theta}[T_1 + T_2] \tag{20}$$

with

$$\begin{aligned}
T_1 &= && C_4(e^{-At} - C_1)^{\frac{1}{B}} \\
T_2 &= T_2' &&= C_2\tfrac{(1-\theta)B}{A-B\rho} \cdot {}_2\tilde{F}_1(t) \cdot e^{(A\theta-\rho)t}(e^{-At} - C_1)^{\theta} \\
T_2 &= T_2'' &&= C_2\tfrac{(1-\theta)B}{A-B\rho} \cdot \tfrac{{}_2\tilde{F}_1(t)}{(1-C_1e^{At})^{-\frac{A\theta-\rho}{A}}} \cdot (e^{-At} - C_1)^{\frac{\rho}{A}} \\
T_2 &= T_2''' &&= C_2\tfrac{(1-\theta)B}{A-B\rho} \cdot \tfrac{{}_2\tilde{F}_1(t)}{\ln(1-C_1e^{At})} \cdot \ln(1 - C_1e^{At})e^{(A\theta-\rho)t}(e^{-At} - C_1)^{\theta} \\
T_2 &= T_2'''' &&= C_2\tfrac{(1-\theta)B}{A-B\rho} \cdot {}_2\tilde{F}_1(t) \cdot e^{-\rho t}(1 - C_1e^{At})^{\theta}
\end{aligned} \tag{21}$$

were we collected the parameters in a proper manner for the following proof. Suppose $C_2, C_4 \neq 0$ and $A - B\rho \neq 0$ in the following.

1. For $A > 0, B > 0$ and $C_1 \leq 0$ firstly we have $\lim_{t\to\infty} T_1 = -C_1^{\frac{1}{B}} = 0$ iff $C_1 = 0$. Secondly we have $\lim_{t\to\infty} \frac{C_1e^{At}}{C_1e^{At}-1} = 1$ and thirdly from *http://dlmf.nist.gov/15.4#ii* we have the following limit theorems for hypergeometric functions:
   (a) $c - a - b > 0$, then $\lim_{z\to1} {}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$
   (b) $c - a - b < 0$, then $\lim_{z\to1} \frac{{}_2F_1(a,b;c;1)}{(1-z)^{c-a-b}} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$
   (c) $c - a - b = 0$, then $\lim_{z\to1} \frac{{}_2F_1(a,b;c;1)}{-\ln(1-z)} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}$

   Altogether we see, that the convergence of the hypergeometric part of (TVC) depends on $sign\{A\theta - \rho\}$ since from equation (19) we obtain $c - a - b = A\theta - \rho$. Together with the representations T_2', in case of $A\theta - \rho > 0$, T_2'' in case of $A\theta - \rho < 0$ and T_2''' in case of $A\theta - \rho = 0$ for T_2, we directly obtain $T_2 \to 0$ iff $C_1 = 0$.

2. $A, B < 0$ and $C_1 < 1$. We directly obtain $\lim_{t\to\infty} T_1 = 0$ and since $\lim_{t\to\infty} \frac{C_1e^{At}}{C_1e^{At}-1} = 0$ in this case, together with ${}_2F_1(a, b; c; 0) = 1$ by using T_2'''' for T_2 we obtain $\lim_{t\to\infty} T_2 = 0$. Thus in this case we do not need $C_1 = 0$ which implies a non linear dependence between consumption and capital.

$\blacksquare$

Corollary 1

1. *For* $A > 0, B > 0$ *and* $C_1 \leq 0$ *we have* $\mathcal{C}(t) = \frac{A}{B}$ *implying*

$$\begin{aligned} k(t) &= \left[C_2 \tfrac{(1-\theta)B}{A-B\rho} + C_4 e^{-(\frac{A}{B}-\rho)t}\right]^{\frac{1}{1-\theta}} \\ c(t) &= \tfrac{A}{B}\left[C_2 \tfrac{(1-\theta)B}{A-B\rho} + C_4 e^{-(\frac{A}{B}-\rho)t}\right]^{\frac{1}{1-\theta}} \end{aligned} \tag{22}$$

2. *For* $A < 0, B < 0$ *and* $C_1 < 1$ *we generally have* $\mathcal{C}(t) = \frac{\frac{A}{B}}{1-C_1 e^{At}}$ *and*

$$\begin{aligned} k(t) &= \left[\tfrac{C_2(1-\theta)B}{A-B\rho} {}_2F_1\left(\tfrac{1}{B}-\theta,1;\tfrac{1}{B}-\tfrac{\rho}{A}+1;\tfrac{C_1 e^{At}}{C_1 e^{At}-1}\right) + \tfrac{C_4}{(e^{At})^{\frac{1}{B}-\frac{\rho}{A}}(1-C_1 e^{At})^{\theta-\frac{1}{B}}}\right]^{\frac{1}{1-\theta}} \\ c(t) &= \tfrac{\frac{A}{B}}{1-C_1 e^{At}}\left[\tfrac{C_2(1-\theta)B}{A-B\rho} {}_2F_1\left(\tfrac{1}{B}-\theta,1;\tfrac{1}{B}-\tfrac{\rho}{A}+1;\tfrac{C_1 e^{At}}{C_1 e^{At}-1}\right) + \tfrac{C_4}{(e^{At})^{\frac{1}{B}-\frac{\rho}{A}}(1-C_1 e^{At})^{\theta-\frac{1}{B}}}\right]^{\frac{1}{1-\theta}} \end{aligned} \tag{23}$$

Proof :
The proof is directly given by proposition 2 and inserting $c(t) = \mathcal{C}(t)k(t)$ for consumption.

$\blacksquare$

In order to embed this calculations in the literature observe that from equation (12) the standard case of Cobb-Douglas production technology and capital's share equals consumer's risk aversion parameter is included in our setup if $A = \frac{1}{\theta}(\delta(1-\theta)+\rho)$ together with $B = 1$ and the endogenous growth extension of Smith (2006) can be found for $0 < A < \frac{1}{\theta}(\delta(1-\theta)+\rho)$ also with $B = 1$. Furthermore inserting the result $\mathcal{C} = \frac{A}{B}$ of proposition 2 for $A, B > 0$ into equation (13) we obtain a standard Bernoulli differential equation with constant coefficients

$$\dot{k} + \mathcal{A}k = \mathcal{B}k^{\theta} \tag{24}$$

with $\mathcal{A}\frac{\frac{A}{B}-\rho}{1-\theta}$ and $\mathcal{B} = C_2$. This is widely used in growth theory especially with the Solow and the Ramsey model. Although this also a standard undergraduate textbook case (i.e. $\mathcal{A} = 0\,\mathcal{B} = 1$ and $\theta = \frac{1}{3}$ Forster (2011)) for non-uniqueness of solutions if $k(0) = 0$ this is often neglected in economic papers. But recently this indeterminacy also discussed in the economic literature Bose (2007) and Hakenes and Irmen (2008).

We end with an explicit solution for $A, B < 0$,where we chose a set of parameters which leads to a compact solution: $A = -1$, $B = -\frac{4}{3}$, $\theta = \frac{1}{4}$, $\rho = \frac{1}{2}$, $\delta = \frac{1}{2}$, $C_2 = 1$ and $C_4 = \sqrt{2}$, which implies

$$ {}_2\tilde{F}_1(t) = {}_2F_1\left({\scriptstyle -1,1;\frac{1}{2};\frac{C_1 e^{At}}{C_1 e^{At}-1}}\right) = \frac{2e^{2t}+2}{2e^{2t}-2} \tag{25} $$

since from $dlmf.nist.org.gov/15.4$ we have

$$ {}_2F_1\left({\scriptstyle -a,a;\frac{1}{2};-z^2}\right) = \frac{1}{2}\left(\left(\sqrt{1+z^2}+z\right)^{2a} + \left(\sqrt{1+z^2}-z\right)^{2a}\right) \tag{26} $$

Altogether we end up with

$$ \begin{aligned} k(t) &= \left[\coth(\tfrac{1}{2}(t+\ln\sqrt{2}))\right]^{\frac{4}{3}} \\ c(t) &= \tfrac{3}{2-e^{-2t}} \cdot \left[\coth(\tfrac{1}{2}(t+\ln\sqrt{2}))\right]^{\frac{4}{3}} \\ \mathcal{C}(t) &= \tfrac{3}{2-e^{-2t}} \end{aligned} \tag{27} $$

In the limit $t \longrightarrow \infty$ we obtain

$$ \begin{aligned} \lim_{t\longrightarrow\infty} k(t) &= k^\infty &= 1 \\ \lim_{t\longrightarrow\infty} c(t) &= c^\infty &= \tfrac{3}{2} \\ \lim_{t\longrightarrow\infty} \mathcal{C}(t) &= \mathcal{C}^\infty &= \tfrac{3}{2} \end{aligned} \tag{28} $$

Graphically this is shown in figure 1

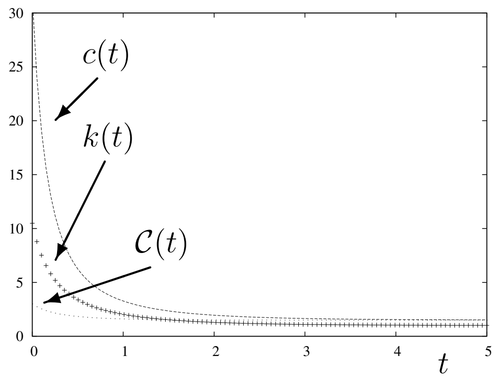

Figure 1:
Evolution in time of capital, consumption and consumption per capital for $A = -1$, $B = -\frac{4}{3}$, $\theta = \frac{1}{4}$, $\rho = \frac{1}{2}$, $\delta = \frac{1}{2}$, $C_2 = 1$ and $C_4 = \sqrt{2}$.

3 Discussion

Our approach closed a small gap in the discussion about closed form solutions in economic growth models. Although the mathematics mainly repeats only some older papers we could show that there is a deeper dependence between standard utility functions and standard technology functions, since our inverse approach showed that CRRA consumer preferences together with logistic consumption per capital growth path consequently requires a Cobb-Douglas part in the production function. Additionally at least for academic purpose we obtain a closed form solution with a non constant ratio of consumption per capital. Finally, we think that with the use of hypergeometric functions, there can be found closed form solutions in a wide range of economic models beyond the Ramsey model like a Sidrauski-type or a Lucas-Uzawa-type model, since firstly hypergeometric functions include many standard functions and secondly via some standard math programs these functions are easily accessible.

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